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EVASION CONDITIONS IN A SECOND-ORDER LINEAR DIFFERENTIAL GAME

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Necessary and sufficient evasion conditions in a second-order linear differential game are derived. This paper is closely related with [1 – 4].

1. We consider the second-order system

$$dx/dt = Ax + u - v \tag{1.1}$$

Here x is a two-dimensional phase vector, A is a constant 2×2 matrix, u and v are the controls of the first and second players respectively. We assume that at any instant t

$$u(t) \in U, \qquad v(t) \in V \tag{1.2}$$

where U is a segment on a plane, not reducing to a point, and V is a bounded closed convex set. The termination of the game means the hitting of system (1.1) onto a certain preassigned point m.

Let us define the notion of evasion. Let the "realization $u(\cdot)$" be a measurable time function $u(t)$, $t_0 \leqslant t < \infty$, satisfying constraint (1.2) for any t and formed by the first player during the game by some method. We take it that when $t \geqslant t_0$ the second player can collide with any realization $u(\cdot)$. The second player is obliged to construct his own control on the feedback principle by means of the discrete scheme $\{v[x], \Delta[x]\}$. The discrete time step $\Delta[x] > 0$ defines the size of the semi-interval $t^* \leqslant t < t^* + \Delta[x[t^*]]$ during which the control v is held constant and depends upon the position $x[t^*]$, where it is chosen in accordance with $v[x]$.

The discrete scheme $\{v[x], \Delta[x]\}$ is said to be admissible if for any intial position x_0 and for any realization $u(\cdot)$ the switching instants of control v cannot tend from the left to a limit t_* not coinciding with the instant at which system (1.1) hits onto point m. By $T[x_0; v[x], \Delta[x], u(\cdot)]$ we denote the time taken by system (1.1) to go from the initial position x_0 to point m under an admissible scheme $\{v[x],\ \Delta[x]\}$ and a realization $u(\cdot)$. Let

$$T[x_0;\ v[x],\ \Delta[x]] = \inf_{u(\cdot)} T[x_0;\ v[x],\ \Delta[x],\ u(\cdot)]$$

where the greatest lower bound is taken over all realizations $u(\cdot)$.

Definition. An evasion is possible in the game if an admissible discrete scheme $\{v^\circ[x],\ \Delta^\circ[x]\}$ exists for which the time $T[x_0;\ v^\circ[x],\ \Delta^\circ[x]] = \infty$ for any initial position $x_0 \neq m$.

We do not examine the case when the set V is a set on a straight line passing through the segment U. The necessary and sufficient evasion conditions for this case follow from well-known results in the theory of differential games [1, 3, 5]. For subsequent convenience we take it that the point m coincides with the origin and that the segment U lies on the x_2-axis.

2. We assume that the set V does not intersect the x_2-axis and, to be specific, lies off the x_2-axis strictly to the right. We choose a neighborhood O of point m such that for any $x \in O$ the set $-Ax + V$ lies strictly to the right of the x_2-axis. Let ξ be an arbitrary ray issuing from point m and directed to the right of the x_2-axis. The maximal collection of straight lines parallel to ray ξ and passing through the set U $(-Ax + V)$ is called the strip $P(\xi)$ $(Q(x,\ \xi))$. For any $x \in O$ we define the sectors $k_1(x)$, $k_2(x)$, $k(x)$, $s(x)$

$$\begin{aligned} k_1(x) &= \{\xi:\ P(\xi) = Q(x,\ \xi)\} \\ k_2(x) &= \{\xi:\ P(\xi) \supset Q(x,\ \xi), \quad P(\xi) \neq Q(x,\ \xi)\} \\ k(x) &= k_1(x) \cup k_2(x) \\ s(x) &= \{\xi:\ P(\xi) \subset Q(x,\ \xi), \quad P(\xi) \neq Q(x,\ \xi)\} \end{aligned}$$

Note that the condition that the sector $k_2(x)$ $(s(x))$ is nonempty at an arbitrary point $x \in O$ implies that it does not degenerate into a ray at this point. To the contrary, if the sector $k_1(x)$ is not empty for some $x \in O$, it consists of one ray at this point.

Theorem 2.1. Let the set V lie strictly to the right of the x_2-axis. For an evasion to be possible it is necessary and sufficient that at least one of the following two conditions be fulfilled: (1) $s(m) \neq \varnothing$, (2) $k_1(m) \neq \varnothing$ and a neighborhood $L \subset O$ of point m exists such that $s(x) \neq \varnothing$ for any $x \in (L \setminus \{m\}) \cap k_1(m)$.

The proof of the theorem follows from the lemmas and a corollary which follow.

Assume that $k_1(m) \neq \varnothing$. By β we denote the straight line on which the ray $k_1(m)$ lies. Suppose that the straight line β is not invariant relative to a transformation A corresponding to matrix A. Then, the set $\gamma = \{x:\ Ax \in \beta\}$ is a straight line passing through point m. The straight line divides the plane X into two halfplanes. That one of them which contains the ray $k_1(m) \setminus \{m\}$ is called Γ. We do not include the straight line γ in the halfplane Γ. By $C(r)$ we denote a circle of radius r with center at point m, imbedded in O. If $k_1(m) \neq \varnothing$ and the straight line β is not invariant, we set $D(r) = C(r) \cap \Gamma$. If $k_1(m) \neq \varnothing$ and the straight line β is invariant, then $D(r) = C(r) \cap \beta$. For $k_1(m) = \varnothing$ we set $D(r) = C(r)$.

Lemma 2.1. Let $k_1(m) \neq \varnothing$. Then a number $r^* > 0$ exists such that either $k(x) \neq \varnothing$ for any $x \in D(r^*)$ or $s(x) \neq \varnothing$ for any $x \in D(r^*)$.

Corollary 2.1. Let $k_1(m) \neq \varnothing$. Then the following two conditions are equivalent: (1) a neighborhood $L \subset O$ of point m exists such that $k(x) \neq \varnothing$ $(s(x) \neq \varnothing)$ for any $x \in (L \setminus \{m\}) \cap k_1(m)$; (2) a number $r^* > 0$ exists such that $k_1(x) \neq \varnothing$ $(s(x) \neq \varnothing)$ for any $x \in D(r^*)$.

Lemma 2.2. Let $k_2(m) \neq \varnothing$ $(s(m) \neq \varnothing)$. Then a number $r^* > 0$ exists such that $k_2(x) \neq \varnothing$ $(s(x) \neq \varnothing)$ for any $x \in D(r^*)$.

Lemma 2.3. If a number $r^* > 0$ exists such that $k(x) \neq \varnothing$ for any $x \in D(r^*)$, evasion is impossible.

Lemma 2.4. If a number $r^* > 0$ exists such that $s(x) \neq \varnothing$ for any $x \in D(r^*)$, evasion is possible.

We note that the proofs of Lemmas 2.2, 2.3 are constructive in nature. Namely, when proving Lemma 2.2 we first construct a certain set η different from $\{m\}$, of initial positions x_0. Next we indicate a method for forming, from any $x_0 \in \eta$ and from any discrete scheme $\{v[x],\ \Delta[x]\}$ a realization $u(\cdot)$ for which the time $T[x_0;\ v[x],\ \Delta[x],\ u(\cdot)] < \vartheta$. Here the number $\vartheta < \infty$ depends neither on $x_0 \in \eta$ not on $\{v[x], \Delta[x]\}$. The proof of Lemma 2.3 is based on the construction of a discrete scheme $\{v^\circ[x],\ \Delta^\circ[x]\}$ for which $T[x_0;\ v^\circ[x],\ \Delta^\circ[x]] = \infty$ for any $x_0 \neq m$.

As an example to Theorem 2.1 we consider the system

$$dx_1/dt = x_2 + u_1 - v_1, \qquad dx_2/dt = u_2 - v_2 \tag{2.1}$$

with the constraints

$$U = \{u : u_1 = 0,\ |u_2| \leqslant 1\}$$
$$V = \{v : 1 \leqslant v_1 \leqslant 3,\ v_2 = \text{const}\}$$

Let O be an open circle of radius $1/2$ with center at point m. We distinguish four cases: (1) $|v_2| < 2$, (2) $v_2 = -2$, (3) $v_2 = 2$, (4) $|v_2| > 2$. These cases are shown in Figs. 1 – 4, respectively. In the first one $k_2(m) \neq \varnothing$, in the second and third $k_1(m) \neq \varnothing$, in the fourth $s(m) \neq \varnothing$. When $v_2 = -2$ $(v_2 = 2)$ the ray $k_1(m)$ is directed below (above) the x_1-axis, and $k_2(x) \neq \varnothing$ $(s(x) \neq \varnothing)$ for any $x \in (O \setminus \{m\}) \cap k_1(m)$. From Theorem 2.1 it follows that evasion is impossible in cases (1), (2), and is possible in cases (3), (4). Let us show immediately how to formulate, in cases (1), (2), a realization $u(\cdot)$ which hinders an evasion, and how to construct, in cases (3), (4), a discrete scheme $\{v^\circ[x],\ \Delta^\circ[x]\}$ which ensures an evasion.

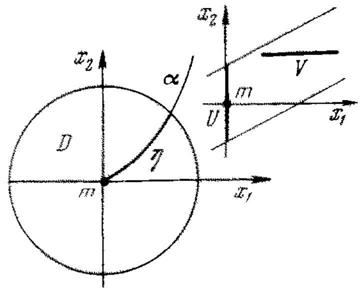

Fig. 1.

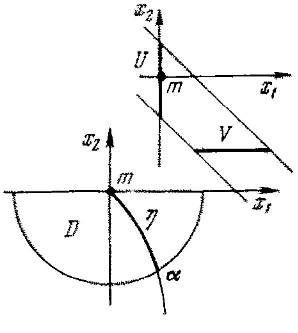

Fig. 2.

1°. We consider cases (1), (2). In case (1) we denote by D an open circle of radius $1/4\,(2 - |v_2|)$ with center at point m (Fig. 1). In case (2) we set $D = O \cap \{x : x_2 < 0\}$ (Fig. 2). If $v_2 \geqslant 1$, we take $v_1^* = 1$. If $v_2 < 1$, we take $v_1^* = 3$. From the point m we trace, in reverse time $\tau = -t$ the motion of system (2.1) under constant $u_2 = 1$ and $v_1 = v_1^*$.

The trajectory of this motion is denoted by α. Let $\eta = (\alpha \cap D) \cup \{m\}$. The curve η satisfies the equation

$$\frac{dx_2}{dx_1} = \frac{v_2 - 1}{v_1{}^* - x_2} \tag{2.2}$$

we define the function

$$u_2[x, v_1] = (x_2 - v_1)\frac{v_2 - 1}{v_1{}^* - x_2} + v_2$$

on the product $O \times V$. By direct verification we convince ourselves that $|u_2[x, v]| \leqslant 1$ in the set $(D \cup \{m\}) \times V$.

We introduce the system

$$dx_1 / dt = x_2 - v_1(t), \qquad dx_2 / dt = u_2[x, v_1(t)] - v_2 \tag{2.3}$$

Here $v_1(t)$ is an arbitrary measurable function satisfying the condition $1 \leqslant v_1(t) \leqslant 3$ for any t. In the set O the motion trajectory of this system is described by Eq. (2.2). Therefore, if $x_0 = x(t_0) \in \eta \setminus \{m\}$, then for $t \geqslant t_0$ the system goes to the left along the curve η and reaches point m in the time

$$\Delta t \leqslant \max \frac{x_{10}}{|x_2 - v_1|} < 1, \ x_{10} < 1/2, \ |x_2| < 1/2, \ 1 \leqslant v_1 \leqslant 3$$

Since $\eta \subset D \cup \{m\}$, the measurable function $u_2(t) = u_2[x(t), v_1(t)]$ satisfies the condition $|u_2(t)| \leqslant 1$ for any t from the transition time interval. Consequently, for any $x_0 \in \eta$ and for any admissible discrete scheme $\{v[x], \Delta[x]\}$ the realization $u(\cdot)$ formed with the aid of the function $u_2[x, v_1]$ yields the result $T[x_0; v[x], \Delta[x], u(\cdot)] < 1$. Consequently, such a realization hinders evasion.

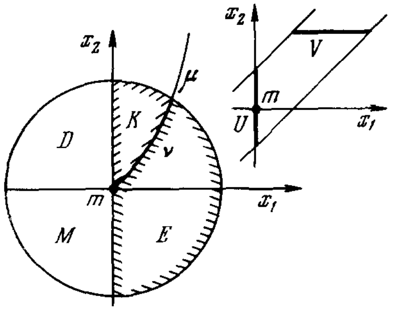

Fig. 3.

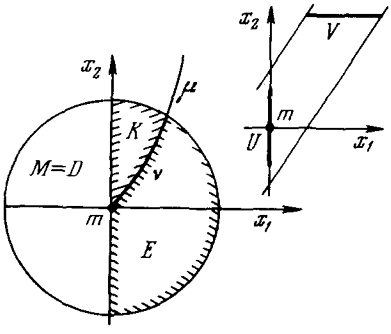

Fig. 4.

2°. We consider cases (3), (4). In case (3) we set $D = O \cap \{x : x_2 > 0\}$ (Fig. 3). In case (4) we denote by D an open circle of radius $r = \min\{1/2, 1/2(|v_2| - 2)\}$ with center at point m (Fig. 4). Note that $s(x) \neq \varnothing$ for any $x \in D$. If $v_2 \geqslant 2$, we take $v_1{}^* = 1$ ($v_{1*} = 3$). If $v_2 < -2$, we take $v_1{}^* = 3$ ($v_{1*} = 1$). From the point m we trace, in reverse time $\tau = -t$ the motion of system (2.1) under constraint $u_2 = 1$ and $v_1 = v_1{}^*$. The trajectory of this motion is denoted by μ. Let $\nu = \mu \cap D$. By M we denote the smallest open circle with center at point m, containing set D within itself. We set $H = M \cap \{x : x_1 > 0\}$ and we let E (K) be the part of set H lying below (above) curve ν. We include the curve ν in K.

Let x_0 be an arbitrary point of E (K). From x_0 we trace the motion of system (2.1) under constant $u_2 = 1$ ($u_2 = -1$) and $v_1 = v_1{}^*$ ($v_1 = v_{1*}$). By $\chi(x_0)$ ($\lambda(x_0)$) we denote the intersection of the trajectory of this motion with the set H. Obviously, the curve $\chi(x_0) \subset E$. Since $s(x) \neq \varnothing$ in D and $\nu \subset D$, we have that for any $x_0 \in K \setminus \nu$ $(x_0 \in \nu)$ the curve $\lambda(x_0)$ $(\lambda(x_0) \setminus \{x_0\})$ together with some sufficiently small neighborhood of itself, lies in K strictly above the curve ν.

We define the discrete scheme $\{v^\circ[x],\ \Delta^\circ[x]\}$

$$v^\circ[x] = \begin{cases} \begin{pmatrix} v_1^* \\ v_2 \end{pmatrix}, & x \in E \\ \begin{pmatrix} v_{1*} \\ v_2 \end{pmatrix}, & x \in K \\ v \in V, & x \in X \setminus H \end{cases} \qquad (\Delta^\circ[x] \equiv \Delta)$$

Here v is any element of V. We select the number $\Delta > 0$ such that for any constant $v \in V$ and for any realization $u(\cdot)$ the system (2.1) cannot pass from the set $X \setminus M$ to point m in the time $\Delta t \leqslant \Delta$. Let us assume that the second player applies the discrete scheme $\{v^\circ[x],\ \Delta^\circ[x]\}$. Let $x_0 = x(t_0) \in (M \setminus \{m\}) \setminus H$. The component $\dot{x}_1 = x_2 - v_1$ of the velocity vector of system (2.1) along the x_1-axis is negative on $O \times V$, and so, whatever be the realization $u(\cdot)$, when $t \geqslant t_0$ the phase point moves to the left and, not hitting onto point m, departs from the boundary of circle M after a finite time.

Let $x_0 = x(t_0) \in E$ (K). Since $u_2 \leqslant 1$ $(u_2 \geqslant -1)$, when $t \geqslant t_0$ the phase point does not go, upto the first instant of going onto the boundary of set H above (below) the curve $\chi(x_0)$ $(\lambda(x_0))$ whatever be the realization $u(\cdot)$. Consequently, the point of first contact with the boundary of set H is different from m and belongs either to that part of the boundary of set H that lies on the boundary of circle M or to that part located on the x_2-axis. In the latter case, by what we said earlier, the system (2.1), by-passing the point m, goes onto the boundary of circle M. Thus, for any $x_0 \in M \setminus \{m\}$ and for any realization $u(\cdot)$ the system (2.1) cannot hit onto point m without first going onto the boundary of set M. From this and from the definition of the number Δ it follows that the time $T[x_0;\ v^\circ[x],\ \Delta^\circ[x]] = \infty$ for any $x_0 \in X \setminus \{m\}$.

3. **Theorem 3.1.** If set V intersects the x_2-axis, evasion is possible.

We omit the proof of this theorem. We remark only that it reduces to the construction of a discrete scheme $\{v^\circ[x],\ \Delta^\circ[x]\}$ for which $T[x_0;\ v^\circ[x],\ \Delta^\circ[x]] = \infty$ for any $x_0 \neq m$. The construction of such a discrete scheme is demonstrated below by an example.

We consider system (2.1) with the constraints

$$U = \{u : u_1 = 0,\ |u_2| \leqslant 1\},$$
$$V = \{v : 0 \leqslant v_1 \leqslant 3,\ v_2 = 0\} \qquad (3.1)$$

We denote the right endpoint of segment V by v^*. We construct a closed acute-angled sector B with vertex at point m, containing strictly within itself the segment $-U + v^*$. We place sector B in a sector C (Fig. 5). Let l denote the maximal modulus of the slope of the generators of sector C. Let O be an open circle with center at point m, for any point x of which the set

$$-\begin{pmatrix} x_2 \\ 0 \end{pmatrix} - U + v^*$$

lies strictly within the sector B. Set $O_1 = O \cap B$ and $O_2 = O \setminus O_1$. We define a

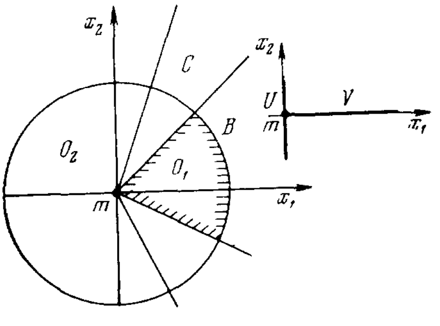

Fig. 5.

discrete scheme $\{v^\circ[x],\ \Delta^\circ[x]\}$. We take

$$v^\circ[x] = \begin{cases} 0, & x \in O_1 \\ v^*, & x \in O_2 \\ v \in V, & x \in X \setminus O \end{cases}$$

Let $\Delta^\circ[x] \equiv \Delta$ in the set $X \setminus O_1$. We choose the number $\Delta > 0$ such that for any constant $v \in V$ and for any realization $u(\cdot)$ the system (2.1) cannot go from the set $X \setminus O$ to point m in the time $\Delta t \leqslant \Delta$. When $x \in O_1 \setminus \{m\}$ we set $\Delta^\circ[x] = \min\{\Delta,\ q(x)\}$, where $q(x)$ is the smallest, with respect to $u(\cdot)$, time for the system under a constant $v = 0$ to go from point x onto the boundary of sector C. We see that the discrete scheme $\{v^\circ[x],\ \Delta^\circ[x]\}$ is admissible. We assume that the second player applies the discrete scheme $\{v^\circ[x],\ \Delta^\circ[x]\}$.

From the definition of circle O it follows that for $v = v^*$ and for any $x \in O_2$, $u \in U$ the velocity vector of system (2.1) is directed toward the exterior of sector B and has a negative component along the x_1-axis, bounded in absolute value. Therefore, if $x_0 = x(t_0) \in O_2$, then whatever be the realization $u(\cdot)$ the phase point moves for $t \geqslant t_0$ in the set O_2 upto the first instant of contact with the boundary of circle O, and consequently, up to this instant cannot hit onto point m.

Let $x_0 = x(t_0) \in O_1 \setminus \{m\}$. We fix an arbitrary realization $u(\cdot)$ and by t^* we denote the first instant that the phase point $x(t)$ goes onto the boundary of the set $O \cap C$. We assume that the instant t^* is finite. Then at this instant the phase point is located either on the boundary of circle O or inside O, but on the boundary of sector C. Let us examine the second possibility. We first show that $x(t^*) \neq m$. Assume the contrary. Then from the definition of the discrete scheme $\{v^\circ[x],\ \Delta^\circ[x]\}$ in set O_1 and from what we have said above regarding the behavior of the phase point when $x_0 \in O_2$, we conclude that the control $v(t) \equiv 0$ is realized on the semi-interval $[t_0,\ t^*)$. Since $x(t) \subset O \cap C$ for any $t \in [t_0,\ t^*)$, we obtain that the modulus of the component of the velocity vector of system (2.1) along the x_1-axis satisfies the estimate

$$|x_1^{\cdot}(t)| = |x_2(t)| < lx_1(t), \qquad t \in [t_0,\ t^*)$$

hence, $x_1(t^*) > 0$. The latter contradicts the assumption that $x(t^*) = m$. Thus the point $x(t^*)$ lies in O on the boundary of sector C and does not coincide with m. Taking into account that t^* is the first instant of contact with the boundary of set $O \cap C$, from the definition of the function $\Delta^\circ[x]$ in O_1 we obtain that on the interval $[t_0,\ t^*]$ we can find an instant t_* for which $x(t_*) \in (O \cap C) \setminus O_1 \subset O_2$ and at which the control v switches from $v = 0$ to $v = v^*$. For $t \geqslant t_*$ the phase point moves in set O_2 up to the instant of reaching the boundary of circle O.

Thus, for any $x_0 \in O \setminus \{m\}$ and for any realization $u(\cdot)$ system (2.1) under constraints (3.1) cannot hit onto point m in a finite time without first reaching the boundary of the open circle O. However, the system cannot go from $X \setminus O$ to the point m in such a way that there would not be any instant t_* on the transition interval for which $x(t_*) \in O \setminus \{m\}$ and at which the control v could switch. This follows from the definition of the function $\Delta^\circ[x]$. Consequently, the time $T[x_0;\ v^\circ[x],\ \Delta^\circ[x]] = \infty$ for any $x_0 \in X \setminus \{m\}$.

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